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Potential symmetries to systems of nonlinear diffusion equations

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Abstract
In this paper, the potential symmetry method is developed to study systems of nonlinear diffusion equations. Potential variables of the systems are introduced through conservation laws; such conservation laws yield equivalent systems-auxiliary systems of PDEs with the given dependent and potential variables as new dependent variables. Lie point symmetries of the auxiliary systems which cannot be projected to the vector fields of the given dependent and independent variables yield potential symmetries of the systems. Classification for systems of nonlinear diffusion equations with two and three components is performed. Symmetry reductions associated with the potential symmetries are presented.

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1. Introduction

It is well known that the group-theoretic methods based on local symmetries or potential symmetries are powerful to seek symmetry reductions and group-invariant solutions of partial differential equations (PDEs) [1–3], to verify whether or not a given system can be linearized by invertible mappings [1], and to construct conservation laws through Noether's theorem [1, 2]. The local symmetries include Lie point symmetries, contact symmetries, Lie Bäcklund symmetries, conditional symmetries and more generally the generalized conditional symmetries. The methods associated with those symmetries have been successfully applied to construct symmetry reductions and exact solutions to a large number of nonlinear PDEs.

Bluman *et al* [1, 4–6] introduced the concept of potential symmetry (or nonlocal symmetry) for a PDE system, say $R\{t, x, \vec{u}\}$ in the case that at least one of the PDEs can be written in a conserved form. If we introduce the potential variables $\vec{v}$ for the PDE system $R\{t, x, \vec{u}\}$ written in a conserved from, as further unknown functions we obtain another system-auxiliary system, say $S\{t, x, \vec{u}, \vec{v}\}$. A Lie point symmetry of $S\{t, x, \vec{u}, \vec{v}\}$ acting on $\{t, x, \vec{u}, \vec{v}\}$ space yields a nonlocal symmetry of the original system $R\{t, x, \vec{u}\}$ if it does not project onto

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a point symmetry acting on $\{t, x, \vec{u}\}$ space. This kind of nonlocal symmetry which is neither Lie point symmetry or Lie–Bäcklund symmetry is said to be the *potential symmetry*.

Recently, the potential symmetries of nonlinear PDEs have been studied in the literatures from several different points of view. Some new physically interesting solutions which cannot be obtained within the framework of Lie's symmetry approach were obtained. Chou and Qu [8] obtained the potential symmetries and the associated similarity solutions for a generalized nonlinear diffusion–convection equation; Gandarias [9], Bluman and Yan [10] obtained a new type of solution to a nonlinear diffusion equation by finding its nonclassical potential symmetries; Senthivelan and Torrisi [13] discussed potential symmetries of a model for reacting mixtures and obtained its new solutions. More interestingly, potential symmetries can be used to construct new conservation laws for some PDEs [11, 12]. But only very few nonlinear PDEs have been found to admit the potential symmetries. So it is of great interest to expand the class of nonlinear PDEs with potential symmetries.

In this paper, we discuss the potential symmetries of the systems of nonlinear diffusion equations

$$u_{it} = \left[\sum_{j=1}^{n} f_{ij}(u_1, \dots, u_n)u_{jx}\right]_x, \qquad i = 1, \dots, n, \tag{1.1}$$

which has a wide range of physical applications (see [14, 15] and references therein). For $n = 1$, its potential symmetries have been discussed by Bluman, Kumei and Reid [4–6], Akhatov, Gazizov and Ibragimov [7]. They showed that the equation

$$u_t = [f(u)u_x]_x \tag{1.2}$$

admits the potential symmetries if and only if

$$f(u) = \frac{1}{u^2 + pu + q} \exp\left(\int^u \frac{r}{s^2 + ps + q} \mathrm{d}s\right).$$

More generally, this result was further extended to the generalized nonlinear diffusion equation

$$u_t = [f(u)(u_x)^n]_x. \tag{1.3}$$

In [8], it was shown that (1.3) admits the potential symmetries if and only if

$$f(u) = \frac{1}{(u^2 + pu + q)^{\frac{3n-1}{2}}} \exp\left(\int^u \frac{r}{s^2 + ps + q} \mathrm{d}s\right).$$

Recently, Gandarias [9] and Bluman and Yan [10] applied the nonclassical method to study the potential symmetries of a nonlinear heat equation and derived some new solutions. A classification of Lie point symmetries to (1.1) with certain functions f_{ij} was performed in [14, 15].

The potential equations (the equations satisfied by potential variables) of system (1.1) for some specific diffusion terms have applications in physical sciences and differential geometry. Let us consider the motion of a space curve in Euclidean geometry governed by

$$\gamma_t = \kappa \mathbf{n}, \tag{1.4}$$

where γ, κ and $\mathbf{n}$ are, respectively, the curve vector, the curvature and normal vector of the curve, which describes the self-induced motion of a superconducting vortex [16]. Alternatively, one may denote the curve in terms of its graph $\gamma = (x, \phi(x,t), \psi(x,t))$, where x is a parameter, $x \in \mathbb{R}^1$. With the graph, the geometric quantities are computed as follows:

$$\mathrm{d}s = g\,\mathrm{d}x, \qquad \mathbf{t} = (1, \phi_x, \psi_x)/g,$$
$$\mathbf{n} = \left(-\tfrac{1}{2}g_x^2, \left(1+\psi_x^2\right)\phi_{xx} - \phi_x\psi_x\psi_{xx}, \left(1+\phi_x^2\right)\psi_{xx} - \phi_x\psi_x\phi_{xx}\right)\big/(g\tilde{g}),$$

$$\begin{aligned}
&\mathbf{b} = (\phi_x\psi_{xx} - \psi_x\phi_{xx}, -\psi_{xx}, \phi_{xx})/\tilde{g}, \\
&\kappa = \tilde{g}/g^3, \qquad \tau = \frac{\phi_{xx}\psi_{xxx} - \psi_{xx}\phi_{xxx}}{\tilde{g}^2}, \\
&g = \sqrt{1+\phi_x^2+\psi_x^2}, \qquad \tilde{g} = \sqrt{\phi_{xx}^2 + \psi_{xx}^2 + (\phi_x\psi_{xx} - \psi_x\phi_{xx})^2},
\end{aligned} \tag{1.5}$$

where s is the arc length, **t**, **n** and **b** are, respectively, the tangent, normal and binormal vectors, κ and τ are, respectively, the curvature and torsion of the curve.

Multiplying (1.4), respectively, by **n** and **b**, and using expressions (1.5), we obtain

$$\begin{aligned}
&\psi_{xx}\phi_t - \phi_{xx}\psi_t = 0, \\
&\left[(1+\psi_x^2)\phi_{xx} - \phi_x\psi_x\psi_{xx}\right]\phi_t + \left[(1+\phi_x^2)\psi_{xx} - \phi_x\psi_x\phi_{xx}\right]\psi_t = -\tilde{g}^2/g^2.
\end{aligned} \tag{1.6}$$

Solving system (1.6) for ϕ_t and ψ_t, we obtain the system

$$\phi_t = \frac{\phi_{xx}}{1+\phi_x^2+\psi_x^2}, \qquad \psi_t = \frac{\psi_{xx}}{1+\phi_x^2+\psi_x^2}. \tag{1.7}$$

Setting $u = \phi_x$, $v = \psi_x$, we arrive at the system

$$u_t = \left(\frac{u_x}{1+u^2+v^2}\right)_x, \qquad v_t = \left(\frac{v_x}{1+u^2+v^2}\right)_x, \tag{1.8}$$

which is a special case of (1.1) with $n = 2$, $f_{11} = f_{22} = 1/(1+u^2+v^2)$, $f_{12} = f_{21} = 0$. It is noted that system (1.7) is a natural generalization of the curve shortening equation [17]:

$$\phi_t = \frac{\phi_{xx}}{1+\phi_x^2}.$$

This equation was shown to admit generalized conditional symmetries and a number of interesting solutions [18]. We point that the group-invariant solutions play the crucial role in the study of asymptotical behaviour and formation of singularities of curves during the curve motion [17].

The outline of this paper is as follows. In section 2, we consider potential symmetries of (1.1) with $n = 2$, $f_{12} = f_{21} = 0$. Similarity reductions to the resulting equations associated with the potential symmetries are presented in section 3. In section 4, we discuss potential symmetries of (1.1) with three components, i.e. $n = 3$, $f_{ij} = 0, i \neq j$. Section 5 contains concluding remarks on this work.

2. Potential symmetries of (1.1) with two components

In this section, we consider the potential symmetries of system (1.1) with $n = 2$, $f_{12} = f_{21} = 0$, that is

$$u_t = (f(u,v)u_x)_x, \qquad v_t = (g(u,v)v_x)_x, \tag{2.1}$$

where $f_v \neq 0$, $g_u \neq 0$. System (2.1) is the simplest one in (1.1). To determine the potential symmetries of system (2.1), let us write (2.1) in the potential form

$$\begin{aligned}
&u = \phi_x, \qquad \phi_t = f(u,v)u_x, \\
&v = \psi_x, \qquad \psi_t = g(u,v)v_x.
\end{aligned} \tag{2.2}$$

We determine transformation groups generated by the vector fields of the form

$$V = \xi\partial_x + \tau\partial_t + \eta_1\partial_u + \eta_2\partial_v + \alpha\partial_\phi + \beta\partial_\psi, \tag{2.3}$$

which are admitted by (2.2). The transformations which cannot be projected to the transformations on $\{t, x, u, v\}$ then induce potential symmetries of equation (2.1). The corresponding potential system to (2.1) reads as the system

$$\phi_t = f(\phi_x, \psi_x)\phi_{xx}, \qquad \psi_t = g(\phi_x, \psi_x)\psi_{xx}. \tag{2.4}$$

It follows from the infinitesimal criterion for invariance of PDEs that (2.2) admits the symmetry group (2.3) if and only if

$$\begin{aligned} &V^{(1)}(u - \phi_x)|_E = 0, \qquad V^{(1)}(\phi_t - f(u, v)u_x)|_E = 0, \\ &V^{(1)}(v - \psi_x)|_E = 0, \qquad V^{(1)}(\psi_t - g(u, v)v_x)|_E = 0, \end{aligned} \tag{2.5}$$

where E denotes the solution set of system (2.2), and

$$V^{(1)} = V + \alpha^x \frac{\partial}{\partial \phi_x} + \beta^x \frac{\partial}{\partial \psi_x} + \alpha^t \frac{\partial}{\partial \phi_t} + \beta^t \frac{\partial}{\partial_{\psi_t}} + \eta_1^x \frac{\partial}{\partial u_x} + \eta_2^x \frac{\partial}{\partial v_x} \tag{2.6}$$

is the first-order prolongation of (2.3), where α^x, β^x, α^t, β^t, η_1^x and η_2^x depend on V, and their expressions can be found in [1–3]. From (2.5), we obtain the determining equations for the infinitesimals $\xi, \tau, \eta_1, \eta_2, \alpha$ and β:

$$\alpha_t - u\xi_t - f(\eta_{1,x} + u\eta_{1,\phi} + v\eta_{1,\psi}) = 0, \tag{2.7a}$$

$$\alpha_u - u\xi_u + f(\tau_x + u\tau_\phi + v\tau_\psi) = 0, \tag{2.7b}$$

$$\alpha_v - u\xi_v = 0, \tag{2.7c}$$

$$f(\alpha_\phi - \tau_t - \eta_{1,u} + \xi_x + v\xi_\psi) - \eta_2 f_v - \eta_1 f_u = 0, \tag{2.7d}$$

$$g(\alpha_\psi - u\xi_\psi) - f\eta_{1,v} = 0, \tag{2.7e}$$

$$\tau_v = 0, \tag{2.7f}$$

$$f\tau_\phi - \xi_u = 0, \tag{2.7g}$$

$$g\tau_\psi - \xi_v = 0, \tag{2.7h}$$

$$\beta_t - v\xi_t - g(\eta_{2,x} + u\eta_{2,\phi} + v\eta_{2,\psi}) = 0, \tag{2.7i}$$

$$\beta_v - v\xi_v + g(\tau_x + u\tau_\phi + v\tau_\psi) = 0, \tag{2.7j}$$

$$\beta_u - v\xi_u = 0, \tag{2.7k}$$

$$f(\beta_\phi - v\xi_\phi) - g\eta_{2,u} = 0, \tag{2.7l}$$

$$g(\beta_\psi - \tau_t - \eta_{2,v} + \xi_x + u\xi_\phi) - \eta_2 g_v - \eta_1 g_u = 0, \tag{2.7m}$$

$$\tau_u = 0, \tag{2.7n}$$

$$\eta_1 - \alpha_x - v\alpha_\psi - u(\alpha_\phi - \xi_x - u\xi_\phi - v\xi_\psi) = 0, \tag{2.7o}$$

$$\eta_2 - \beta_x - u\beta_\phi + v(\xi_x + u\xi_\phi + v\xi_\psi - \beta_\psi) = 0, \tag{2.7p}$$

$$\alpha_u - u\xi_u - f(\tau_x + u\tau_\phi + v\tau_\psi) = 0, \tag{2.7q}$$

$$\alpha_v - v\xi_v - g(\tau_x + u\tau_\phi + v\tau_\psi) = 0. \tag{2.7r}$$

To solve system (2.7), we first note that $\tau_u = \tau_v = 0$, and it follows from (2.7b), (2.7k), (2.7q), (2.7r) that $\tau_x = \tau_\phi = \tau_\psi = 0$. From (2.7$g$) and (2.7$h$), we also have

$$\xi_u = \xi_v = 0.$$

In view of (2.7b), (2.7c) and (2.7j), (2.7k), we have

$$\alpha_u = \alpha_v = \beta_u = \beta_v = 0.$$

Solving (2.7o) and (2.7p), one gets

$$\begin{aligned} \eta_1 &= -\xi_\phi u^2 - \xi_\psi uv + \alpha_\psi v + (\alpha_\phi - \xi_x)u + \alpha_x, \\ \eta_2 &= -\xi_\phi uv - \xi_\psi v^2 + \beta_\phi u + (\beta_\psi - \xi_x)v + \beta_x. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.7e) and (2.7l), we arrive at $f = g$, otherwise f does not depend on v, or g does not depend on u. This contradicts our assumption: $f_v \neq 0$ and $g_u \neq 0$. To proceed, we consider the following cases:

Case 1. $\xi_t \neq 0$. From (2.7a) and (2.7i), we obtain

$$\begin{aligned} f = {} & (\alpha_t - \xi_t u)[-\xi_{\phi\phi}u^3 - 2\xi_{\phi\psi}u^2 v + (\alpha_{\phi\phi} - 2\xi_{x\phi})u^2 - \xi_{\psi\psi}uv^2 \\ & + \alpha_{\psi\psi}v^2 + 2(\alpha_{\phi\psi} - \xi_{x\psi})uv + 2\alpha_{x\psi}v + (2\alpha_{x\phi} - \xi_{xx})u + \alpha_{xx}]^{-1}, \\ g = {} & (\beta_t - \xi_t v)[-\xi_{\phi\phi}u^2 v - 2\xi_{\phi\psi}uv^2 - \xi_{\psi\psi}v^3 + \beta_{\phi\phi}u^2 + 2(\beta_{\phi\psi} - \xi_{x\phi})uv \\ & + (\beta_{\psi\psi} - 2\xi_{x\psi})v^2 + 2\beta_{x\phi}u + (2\beta_{x\psi} - \xi_{xx})v + \beta_{xx}]^{-1}. \end{aligned}$$

It follows from the representations for f and g, and noting that $f = g$, we deduce

$$f = g = [a_1 u^2 + 2b_1 uv + c_1 v^2 + 2a_2 u + 2b_2 v + c_2]^{-1}, \tag{2.9}$$

where a_i, b_i and c_i, $i = 1, 2$ are some constants.

After the suitable translation, dilatation or rotation for u and v, $f = g$ is reduced to one of the followings:

For $|a_1| + |b_1| + |c_1| \neq 0$, $b_1^2 - a_1 c_1 \neq 0$.

$$f = g = [u^2 \pm v^2 + a]^{-1}, \qquad a = \text{Const}. \tag{2.10}$$

For $a_1 = b_1 = c_1 = 0$.

$$f = g = (u + v)^{-1}. \tag{2.11}$$

For $|a_1| + |b_1| + |c_1| \neq 0$, $b_1^2 - a_1 c_1 = 0$, $I = 0$.

$$f = g = [(u + v)^2 + a]^{-1}, \qquad a = \text{Const}, \tag{2.12}$$

where

$$I = \begin{vmatrix} a_1 & b_1 & a_2 \\ b_1 & c_1 & b_2 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

For $|a_1| + |b_1| + |c_1| \neq 0$, $b_1^2 - a_1 c_1 = 0$, $I \neq 0$.

$$f = g = (u^2 + v)^{-1}. \tag{2.13}$$

Case 2. $\xi_t = 0$, $\alpha_t \neq 0$, $\beta_t \neq 0$. Using (2.7a) and (2.7i), we deduce

$$\begin{aligned} f = {} & \alpha_t[-\xi_{\phi\phi}u^3 - 2\xi_{\phi\psi}u^2 v + (\alpha_{\phi\phi} - 2\xi_{x\phi})u^2 - \xi_{\psi\psi}uv^2 \\ & + \alpha_{\psi\psi}v^2 + 2(\alpha_{\phi\psi} - \xi_{x\psi})uv + 2\alpha_{x\psi}v + (2\alpha_{x\phi} - \xi_{xx})u + \alpha_{xx}]^{-1}, \\ g = {} & \beta_t[-\xi_{\phi\phi}u^2 v - 2\xi_{\phi\psi}uv^2 - \xi_{\psi\psi}v^3 + \beta_{\phi\phi}u^2 + 2(\beta_{\phi\psi} - \xi_{x\phi})uv \\ & + (\beta_{\psi\psi} - 2\xi_{x\psi})v^2 + 2\beta_{x\phi}u + (2\beta_{x\psi} - \xi_{xx})v + \beta_{xx}]^{-1}. \end{aligned}$$

Noting $f = g$, and that the coefficients of u^3, uv^2 and v^2u in f and that of v^3, uv^2 and v^2u in g, we must have

$$\xi_{\phi\phi} = \xi_{\psi\psi} = \xi_{\phi\psi} = 0,$$

which implies that f and g are given by (2.9).

Case 3. $\xi_t = 0, \alpha_t \neq 0, \beta_t = 0$. It follows from (2.7*a*) and (2.7*i*) that f is given by

$$f = \alpha_t[-\xi_{\phi\phi}u^3 - 2\xi_{\phi\psi}u^2v + (\alpha_{\phi\phi} - 2\xi_{x\phi})u^2 - \xi_{\psi\psi}uv^2 + \alpha_{\psi\psi}v^2 + 2(\alpha_{\phi\psi} - \xi_{x\psi})uv + 2\alpha_{x\psi}v + (2\alpha_{x\phi} - \xi_{xx})u + \alpha_{xx}]^{-1}, \tag{2.14}$$

and the infinitesimals ξ, ϕ and ψ satisfy

$$-\xi_{\phi\phi}u^2v - 2\xi_{\phi\psi}uv^2 - \xi_{\psi\psi}v^3 + \beta_{\phi\phi}u^2 + 2(\beta_{\phi\psi} - \xi_{x\phi})uv + (\beta_{\psi\psi} - 2\xi_{x\psi})v^2 + 2\beta_{x\phi}u + (2\beta_{x\psi} - \xi_{xx})v + \beta_{xx} = 0. \tag{2.15}$$

It implies that ξ must satisfy

$$\xi_{\phi\phi} = \xi_{\psi\psi} = \xi_{\phi\psi} = 0.$$

Thus (2.14) implies that f and g are given by (2.9).

Case 4. $\xi_t = 0, \alpha_t = 0, \beta_t \neq 0$. The analysis for this case is the same as for the case (3), and we arrive at the same result.

Case 5. $\alpha_t = \xi_t = \beta_t = 0$. Substituting (2.8) into (2.7*a*), (2.7*i*) and (2.7*d*), we obtain

$$\begin{aligned} &-\xi_{\phi\phi}u^3 - 2\xi_{\phi\psi}u^2v - \xi_{\psi\psi}uv^2 + (\alpha_{\phi\phi} - 2\xi_{x\phi})u^2 + \alpha_{\psi\psi}v^2 + 2(\alpha_{\phi\psi} - \xi_{x\psi})uv \\ &\qquad + 2\alpha_{x\psi}v + (2\alpha_{x\phi} - \xi_{xx})u + \alpha_{xx} = 0, \\ &\qquad - \xi_{\phi\phi}u^2v - 2\xi_{\phi\psi}uv^2 - \xi_{\psi\psi}v^3 + \beta_{\phi\phi}u^2 + 2(\beta_{\phi\psi} - \xi_{x\phi})uv \\ &\qquad + (\beta_{\psi\psi} - 2\xi_{x\psi})v^2 + 2\beta_{x\phi}u + (2\beta_{x\psi} - \xi_{xx})v + \beta_{xx} = 0. \end{aligned}$$

Noting that α, β and ξ do not depend on u and v, we arrive at

$$\begin{aligned} &\xi_{\phi\phi} = \xi_{\phi\psi} = \alpha_{\phi\phi} - 2\xi_{x\phi} = \xi_{\psi\psi} = \alpha_{\psi\psi} = 0, \\ &\alpha_{\phi\psi} - \xi_{x\psi} = \alpha_{x\psi} = 2\alpha_{x\phi} - \xi_{xx} = \alpha_{xx} = 0, \\ &\beta_{\phi\phi} = \beta_{\phi\psi} - \xi_{x\phi} = \beta_{\psi\psi} - 2\xi_{x\psi} = 0, \\ &\beta_{x\phi} = 2\beta_{x\psi} - \xi_{xx} = \beta_{xx} = 0. \end{aligned} \tag{2.16}$$

It follows from (2.7*d*) that f satisfies

$$\begin{aligned} &[-\xi_\phi u^2 - \xi_\psi uv + \alpha_\psi v + (\alpha_\phi - \xi_x)u + \alpha_x]f_u + \\ &[-\xi_\phi uv - \xi_\psi v^2 + \beta_\phi u + (\beta_\psi - \xi_x)v + \beta_x]f_v + (2\xi_\phi u + 2\xi_\psi v + 2\xi_x - \tau_t)f = 0. \end{aligned} \tag{2.17}$$

Noting that system (2.3) is invariant under the transformations

$$\tilde{u} = u + \epsilon_1, \qquad \tilde{v} = v + \epsilon_2, \qquad \tilde{\phi} = \phi + \epsilon_1 x, \qquad \tilde{\psi} = \psi + \epsilon_2 x.$$

Solving system (2.16) and using (2.17), we obtain the infinitesimals given by

$$\begin{aligned} &\xi = a\phi + b\psi, \qquad \tau = ct, \qquad \alpha = Ax + B\phi + C\psi, \\ &\beta = Jx + K\phi + L\psi, \qquad \eta_1 = A + Bu + Cv - au^2 - buv, \\ &\eta_2 = J + Ku + Lv - auv - bv^2. \end{aligned}$$

Thus, we have proved the following result.

Theorem 2.1. *System (2.1) admits the potential symmetries if and only if $f = g$ is given by (2.10)–(2.13) or $f = g$ satisfies the following first-order PDE:*

$$(A + Bu + Cv - au^2 - buv)f_u + (J + Ku + Lv - auv - bv^2)f_v = (2au + 2bv - c)f \tag{2.18}$$

up to the translation and dilatation for u and v. The corresponding vector fields of Lie point symmetries to (2.2) with (2.10)–(2.13) and (2.18) are given as follows. For f given by (2.10) with $a \neq 0$,

$$
\begin{aligned}
&V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi,\\
&V_5 = \phi\partial_x - ax\partial_\phi - uv\partial_v - (a+u^2)\partial_u,\\
&V_6 = \psi\partial_x \mp ax\partial_\psi - uv\partial_u \mp (a \pm v^2)\partial_v,\\
&V_7 = \psi\partial_\phi \mp \phi\partial_\psi + v\partial_u \mp u\partial_v,\\
&V_8 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi.
\end{aligned}
\tag{2.19}
$$

For f given by (2.10) with $a = 0$,

$$
\begin{aligned}
&V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi,\\
&X_5 = \phi\partial_x - uv\partial_v - u^2\partial_u,\\
&X_6 = \psi\partial_x - uv\partial_u - v^2\partial_v,\\
&V_7 = \psi\partial_\phi \mp \phi\partial_\psi + v\partial_u \mp u\partial_v,\\
&V_8 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi,\\
&V_9 = v\partial_v + u\partial_u - x\partial_x,\\
&V_{10} = (\phi^2 \pm \psi^2 + 2t)\partial_x - 2(u\phi \pm v\psi)(u\partial_u + v\partial_v).
\end{aligned}
\tag{2.20}
$$

For f given by (2.11),

$$
\begin{aligned}
&V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi,\\
&W_1 = t\partial_t + x\partial_x - u\partial_u - v\partial_v,\\
&V_8 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi,\\
&W_2 = x(u+v)(\partial_u - \partial_v) + x(\phi+\psi)(\partial_\phi - \partial_\psi) + (\phi+\psi)(\partial_u - \partial_v) + 2t(\partial_\phi - \partial_\psi),\\
&W_3 = \partial_u - \partial_v + x(\partial_\phi - \partial_\psi).
\end{aligned}
\tag{2.21}
$$

For f given by (2.12) with $a \neq 0$,

$$
\begin{aligned}
&V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi,\\
&V_8 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi,\\
&W_3 = \partial_u - \partial_v + x(\partial_\phi - \partial_\psi),\\
&W_4 = (\phi+\psi)(\partial_\phi - \partial_\psi) + (u+v)(\partial_u - \partial_v),\\
&W_5 = -(u+v)(u\partial_u + v\partial_v) + \frac{a}{2}(\partial_u + \partial_v) - \frac{a}{2}x(\partial_\phi + \partial_\psi).
\end{aligned}
\tag{2.22}
$$

For f given by (2.12) with $a = 0$, the symmetries of (2.2) are

$$
\begin{aligned}
&V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi,\\
&V_8 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi,\\
&W_3 = \partial_u - \partial_v + x(\partial_\phi - \partial_\psi),\\
&W_6 = x\partial_x - u\partial_u - v\partial_v,\\
&W_7 = t^2\partial_t - \left(\tfrac{1}{4}\lambda^2 + \tfrac{1}{2}t\right)x\partial_x + \left[t\lambda - \left(\tfrac{1}{4}\lambda^2 + \tfrac{1}{2}t\right)\phi\right]\partial_\phi + \left[t\lambda - \left(\tfrac{1}{4}\lambda^2 + \tfrac{1}{2}t\right)\psi\right]\partial_\psi\\
&\qquad + \left(\tfrac{3}{2}t + \tfrac{1}{2}\lambda(\phi+\psi) + \tfrac{1}{4}\lambda^2 + \tfrac{1}{2}x\lambda v\right)(u+v)(\partial_u + \partial_v),\\
&W_8 = \chi\partial_x - (u+v)\chi_\lambda(u\partial_u + v\partial_v),\\
&W_9 = \kappa(\partial_\phi - \partial_\psi) + \kappa_\lambda(u\partial_u - v\partial_v),
\end{aligned}
\tag{2.23}
$$

where χ and κ are the functions of t and $\lambda = \phi + \psi$, and they satisfy the heat equation $h_t = h_{\lambda\lambda}$. Noting that χ and κ in (2.23) satisfy the heat equation, so the system admits an infinite-dimensional symmetry groups. According to the theory of invertible mapping between linear and nonlinear equations [1], the system can be linearized. Indeed setting $w = u + v$, then w satisfies the following nonlinear diffusion equation:

$$w_t = (w^{-2}w_x)_x. \tag{2.24}$$

It can be linearized by a hodograph transformation.

For f given by (2.13),

$$\begin{aligned} &V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi, \\ &W_{10} = x\partial_x - \psi\partial_\psi - u\partial_u - 2v\partial_v, \\ &W_{11} = x\partial_\phi - 2\phi\partial_\psi + \partial_u - 2u\partial_v, \\ &W_{12} = 2t\partial_t + u\partial_u + 2v\partial_v + \phi\partial_\phi + 2\psi\partial_\psi, \\ &W_{13} = -2\phi\partial_x + \psi\partial_\phi + (v + 2u^2)\partial_u + 2uv\partial_v. \end{aligned} \tag{2.25}$$

For f satisfying (2.18),

$$\begin{aligned} &V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi, \\ &V_8 = \phi\partial_\phi + \psi\partial_\psi + x\partial_x + 2t\partial_t. \\ &V_{11} = (a\phi + b\psi)\partial_x + ct\partial_t + (Ax + B\phi + C\psi)\partial_\phi + (Jx + K\phi + L\psi)\partial_\psi \\ &\qquad + (A + Bu + Cv - au^2 - buv)\partial_u + (J + Ku + Lv - auv - bv^2)\partial_v. \end{aligned}$$

It is worth noting that V_5, V_6, X_5, X_6, V_{10} and V_{11} are potential symmetries of (2.1) since they cannot be projected to the vector fields on $\{t, x, u, v\}$ space. So those vectors cannot be obtained from the Lie point symmetry method. It is interesting to compare the result with that for the single nonlinear diffusion equation, i.e. system (1.1) with $n = 1$ [1].

It seems impossible to derive all solutions of the first-order PDE (2.18). But in the case of $a = K = 0$, with loss of generality we put $L = 0$ by the translation for v, we obtain its special solutions given as follows:

(1) $J = \mu^2$, $B \neq \pm\mu$.

$$f = g = (\mu^2 - v^2)^{-1}\left(\frac{\mu - v}{\mu + v}\right)^{\frac{c}{2\mu}} \tilde{f}(\lambda),$$

with

$$\lambda = \frac{u + \frac{A+CB}{B^2-\mu^2}v + \frac{AB+C\mu^2}{B^2-\mu^2}}{(\mu^2 - v^2)^{\frac{1}{2}}\left(\frac{\mu+a}{\mu-v}\right)^{\frac{B}{2\mu}}},$$

where and hereafter $\tilde{f}$ is an arbitrary function of the indicated variable.

(2) $J = \mu^2$, $B = \mu$.

$$f = g = (\mu^2 - v^2)^{-1}\left(\frac{\mu - v}{\mu + v}\right)^{\frac{c}{2\mu}} \tilde{f}(\lambda),$$

with

$$\lambda = \frac{2\mu u + A - \mu C}{2\mu(v + \mu)} - \frac{A + \mu C}{4\mu^2} \ln\frac{v + \mu}{v - \mu}.$$

(3) $J = \mu^2$, $B = -\mu$.

$$f = g = (\mu^2 - v^2)^{-1}\left(\frac{\mu - v}{\mu + v}\right)^{\frac{c}{2\mu}} \tilde{f}(\lambda),$$

with

$$\lambda = \frac{2\mu u - A - \mu C}{2\mu(v - \mu)} + \frac{A - \mu C}{4\mu^2} \ln\frac{v + \mu}{v - \mu}.$$

(4) $J = -\mu^2$.

$$f = g = \frac{e^{\frac{c}{\mu}\arctan\frac{v}{\mu}}}{\mu^2 + v^2}\tilde{f}(\lambda),$$

with

$$\lambda = \frac{u + \frac{A+CB}{B^2+\mu^2}v + \frac{AB-C\mu^2}{B^2+\mu^2}}{(\mu^2+v^2)^{\frac{1}{2}}}\,e^{\frac{B}{\mu}\arctan\frac{v}{\mu}}.$$

(5) $J = 0$, $B \neq 0$.

$$f = g = \frac{1}{v^2}e^{-\frac{c}{v}}\tilde{f}(\lambda),$$

with

$$\lambda = \frac{B^2u + (A + BC)v + AB}{B^2ve^{\frac{B}{v}}}.$$

(6) $J = 0$, $B = 0$.

$$f = g = \frac{1}{v^2}e^{-\frac{c}{v}}\tilde{f}(\lambda),$$

with

$$\lambda = \frac{2uv - 2Cv - A}{2v^2}.$$

(7) $A = B = C = J = K = L = 0$.

$$f = g = u^{-2}e^{-\frac{c}{au+bv}}\tilde{f}\left(\frac{v}{u}\right).$$

(8) $a, b \neq 0$, $C = B = L = 0$, $C + K = 0$, $A = -a\mu$, $J = -b\mu$.

$$f = g = \frac{1}{u^2 + v^2 + \mu}.$$

(9) $a, b, c \neq 0$, $B = L = c/4$, $C + K = 0$, $A = J = 0$.

$$f = g = \frac{1}{u^2 + v^2}.$$

The last two cases are included in (2.10).

3. Symmetry reductions of systems (2.2) with (2.10)

In this section, we study symmetry reductions of system (2.10) in terms of the symmetries (2.19) and (2.20). To obtain non-equivalent symmetry reductions of the systems, one needs to construct an optimal system for the symmetry groups admitted by the systems. There are several ways to construct the optimal system for a given Lie group; discussion on this topic can be found in [2, 3, 19–21].

First we discuss the symmetry reductions of the system

$$u_t = \left(\frac{u_x}{u^2 + v^2}\right)_x, \qquad v_t = \left(\frac{v_x}{u^2 + v^2}\right)_x. \tag{3.1}$$

By introducing the potential variables, its auxiliary system is

$$\begin{aligned} u &= \phi_x, \qquad \phi_t = \frac{u_x}{u^2 + v^2} \\ v &= \psi_x, \qquad \psi_t = \frac{v_x}{u^2 + v^2}. \end{aligned} \tag{3.2}$$

One may show that an optimal system for the symmetry algebras of system (3.2) is spanned by the vector fields

$$
\begin{aligned}
&v_1 = V_9 + c_1 V_8 + c_2 V_7 (c_1 \neq 0), \qquad v_2 = V_9 + c_2 V_7, \qquad v_3 = V_9 + V_1 + c_1 V_7,\\
&v_4 = V_9 - V_1 + c_1 V_7, \qquad v_5 = V_9 + V_1 + c_1 V_4 (c_1 \geqslant 0), \qquad v_6 = V_9 - V_1 + c_1 V_4 (c_1 \geqslant 0),\\
&v_7 = V_9 + V_4, \qquad v_8 = V_9 - V_4, \qquad v_9 = V_8 + c_1 V_7, \qquad v_{10} = V_8 + X_6,\\
&v_{11} = V_8 - X_6, \qquad v_{12} = V_{10} + V_1 + c_1 V_7, \qquad v_{13} = V_{10} - V_1 + c_1 V_7,\\
&v_{14} = V_{10} + c_1 V_7, \qquad v_{15} = V_7 + V_1, \qquad v_{16} = V_7 - V_1, \qquad v_{17} = V_4 + V_1 + c_1 X_5,\\
&v_{18} = V_4 - V_1 + c_1 X_5, \qquad v_{19} = V_4 + X_5, \qquad v_{20} = V_4 - V_5, \qquad v_{21} = X_6 + V_1,\\
&v_{22} = X_6 - V_1, \qquad v_{23} = X_6, \qquad v_{24} = V_1, \qquad v_{25} = V_2,
\end{aligned} \tag{3.3}
$$

where V_i, X_5 and X_6 are given in (2.20). Note that each vector with the constants c_1 or c_2 contains infinitely many elements.

For $V_8 + c_1 X_6$ ($c_1 \neq 0$), the invariants are

$$
z = \frac{x}{\psi} - \frac{c_1}{2}\ln t, \qquad t^{-\frac{1}{2}}\phi, \qquad t^{-\frac{1}{2}}\psi, \qquad \frac{v}{u}, \qquad \frac{1}{v} - \frac{c_1}{2}\ln t.
$$

This leads to the symmetry reduction of (3.2) given by an implicit form

$$
\begin{aligned}
&\phi = t^{\frac{1}{2}} g(z), \qquad \psi = t^{\frac{1}{2}} h(z), \qquad z = \frac{x}{\psi} - \frac{c_1}{2}\ln t,\\
&u = m(z) v, \qquad v = \frac{1}{\frac{c_1}{2}\ln t + f(z)}.
\end{aligned} \tag{3.4}
$$

The substitution of (3.4) into (3.2) yields the following system for $f(z)$, $g(z)$, $h(z)$ and $m(z)$:

$$
\begin{aligned}
&g'' - \frac{2g'h'}{h} = \frac{1}{2}(g - c_1 g')(g'^2 + h'^2),\\
&h'' - \frac{2h'^2}{h} = \frac{1}{2}(h - c_1 h')(g'^2 + h'^2),\\
&m = \frac{g'}{h'}, \qquad f = z + \frac{h}{h'}.
\end{aligned}
$$

For $V_{10} + c_2 V_1 + c_1 V_7 (c_2 \neq 0)$, the invariants are

$$
\begin{aligned}
&z = x - \frac{t}{c_2}[(\phi^2 + \psi^2) + t], \qquad \frac{c_2\sqrt{u^2 + v^2}}{c_2 - 2(u\phi + v\psi)t}, \qquad \sqrt{\phi^2 + \psi^2},\\
&\arctan\frac{u}{v} - \frac{c_1}{c_2}t, \qquad \arctan\frac{\psi}{\phi} + \frac{c_1}{c_2}t.
\end{aligned}
$$

This leads to a symmetry reduction of (3.2):

$$
\begin{aligned}
&\phi = -g(z)\cos\lambda, \qquad \psi = g(z)\sin\lambda, \qquad \lambda = \frac{c_1}{c_2}t + h(z),\\
&u = \frac{\sin\left[h(z) + \frac{c_1}{c_2}t\right]}{\frac{2}{c_2}t g(z)\sin m(z) + f(z)}, \qquad v = \frac{\cos\left[h(z) + \frac{c_1}{c_2}t\right]}{\frac{2}{c_2}t g(z)\sin m(z) + f(z)}.
\end{aligned} \tag{3.5}
$$

Substituting (3.5) into (3.2) gives the following system for $f(z)$, $g(z)$, $h(z)$ and $m(z)$:

$$
\begin{aligned}
&g'' - g h'^2 = -\frac{1}{b} g^2 g'[g'^2 + (gh')^2],\\
&g h'' + 2g'h' = \frac{1}{b} g(c_1 - g^2 h')[g'^2 + (gh')^2],\\
&f = \arcsin\frac{g'}{\sqrt{g'^2 + (gh')^2}}, \qquad m = \frac{1}{\sqrt{g'^2 + (gh')^2}}.
\end{aligned}
$$

For $V_{10}+c_1V_7$, the invariants are

$$z=t,\qquad \sqrt{\phi^2+\psi^2},\qquad \arctan\frac{u}{v}-\frac{c_1x}{\phi^2+\psi^2+2t},$$
$$\arctan\frac{\psi}{\phi}+\frac{c_1x}{\phi^2+\psi^2+2t},\qquad \frac{1}{\sqrt{u^2+v^2}}\left[1-\frac{2(u\phi+v\psi)}{\phi^2+\psi^2+2t}\right].$$

This leads to the symmetry reduction of (3.2)

$$\phi=-g(t)\cos\lambda,\qquad \psi=g(t)\sin\lambda,\qquad \lambda=\frac{c_1x}{2t+g^2}+h(t),$$
$$u=\frac{\sin\left[g(t)-f(t)+\frac{c_1x}{2t+\phi^2+\psi^2}\right]}{\frac{2g}{2t+g^2}\sin f(t)+m(t)},\qquad v=\frac{\cos\left[g(t)-f(t)+\frac{c_1x}{2t+\phi^2+\psi^2}\right]}{\frac{2g}{2t+g^2}\sin f(t)+m(t)}.$$

The associated group invariant solution to (3.2) is given by

$$u=\frac{c_1}{2t_0}\sqrt{2(t_0-t)}\sin\left(\frac{c_1x}{2t_0}\right),\qquad v=\frac{c_1}{2t_0}\sqrt{2(t_0-t)}\cos\left(\frac{c_1x}{2t_0}\right),$$
$$\phi=-\sqrt{2(t_0-t)}\cos\left(\frac{c_1x}{2t_0}\right),\qquad \psi=\sqrt{2(t_0-t)}\sin\left(\frac{c_1x}{2t_0}\right),$$

where $t_0>0$ is a constant.

For $V_4+c_2V_1+c_1X_5$, $(c_1,c_2\neq 0)$, the invariants are

$$z=x-\frac{c_1}{c_2}t\phi,\qquad \phi,\qquad \frac{1}{u}-\frac{c_1}{c_2}t,\qquad \psi-\frac{t}{c_2},\qquad \frac{v}{u}.$$

This leads to a symmetry reduction of (3.2):

$$\begin{aligned}&\phi=h(z), && \psi=\frac{t}{c_2}+g(z), && z=x-\frac{c_1}{c_2}t\phi\\ &u=\frac{1}{\frac{c_1}{c_2}t+f(z)}, && v=\frac{m(z)}{\frac{c_1}{c_2}t+f(z)}.\end{aligned}\tag{3.6}$$

The substitution of (3.6) into (3.2) yields the following system for $f(z)$, $g(z)$, $h(z)$ and $m(z)$:

$$h''=-\frac{c_1}{c_2}hh'(h'^2+g'^2),$$
$$g''=\frac{1}{c_2}(1-c_1hg')(h'^2+g'^2),$$
$$f=\frac{1}{h'},\qquad m=\frac{g'}{h'}.$$

One may show that an optimal system of the symmetries of the system

$$u_t=\left(\frac{u_x}{1+u^2+v^2}\right)_x,\qquad v_t=\left(\frac{v_x}{1+u^2+v^2}\right)_x\tag{3.7}$$

is given by

$$\begin{aligned}&\omega_1=V_1, && \omega_2=V_2, && \omega_3=V_1+c_1V_2, && \omega_4=V_5,\\ &\omega_5=V_1+V_5, && \omega_6=V_4+V_5, && \omega_7=V_5+V_1+c_1V_4,\\ &\omega_8=V_8+c_1V_5, && \omega_9=V_8+V_2+c_1V_5.\end{aligned}\tag{3.8}$$

We now obtain nontrivial symmetry reductions associated with the elements in the optimal system (3.8).

For ω_5, invariants are $z=\sqrt{x^2+\phi^2}$, u, v, $\arcsin(x/z)-t$ and ψ. So the group-invariant solutions are given by

$$\phi=g(z)\cos(t+h(z)),\qquad x=g(z)\sin(t+h(z)),\qquad z=\psi,$$
$$u=\tan(\lambda+m(z)),\qquad v=f(z)\sec(\lambda+m(z)),$$

where $\lambda=t+h(z)$, and $f(z)$, $g(z)$, $h(z)$ and $m(z)$ satisfy

$$g''=gh'^2,\qquad gh''+2g'h'=g(1+g'^2+g^2h'^2),$$
$$f(z)=\frac{g'}{\sqrt{1+(zh')^2}},\qquad m(z)=\frac{zh'}{\sqrt{1+(zh')^2}}.$$

For ω_6, invariants are

$$z=t,\qquad \sqrt{x^2+\phi^2},\qquad \arctan\frac{x}{\phi}-t\quad \frac{v}{\sqrt{1+u^2}},\qquad \arctan u-t.$$

We obtain the group-invariant solutions given by

$$\psi=\arcsin\frac{x}{g}-h(t),\qquad \phi=g(z)\cos\lambda,\qquad x=g(z)\sin\lambda,$$
$$u=-\tan[\lambda+m(z)],\qquad v=f(z)\sec[\lambda+m(z)],$$

where $\lambda=h(t)+\psi$, and $f(z)$, $g(z)$, $h(z)$ and $m(z)$ satisfy

$$g'=-\frac{g}{1+g^2},\qquad m=h'=0,\qquad f=\frac{1}{g}.$$

For ω_7, invariants are

$$z=\psi-c_1t,\qquad \sqrt{x^2+\phi^2},\qquad \frac{v}{\sqrt{1+u^2}},\qquad \arcsin\frac{x}{\phi}-\psi,\qquad \arctan u+\psi.$$

So the group-invariant solutions are given by

$$\psi=z+c_1t,\qquad \phi=g(z)\cos\lambda,\qquad x=g(z)\sin\lambda,$$
$$u=-\tan(\lambda+m(z)),\qquad v=f(z)\sec\lambda,\qquad \lambda=t+h(z).$$

Substituting it into system (3.7) implies $f(z)$, $g(z)$, $h(z)$ and $m(z)$ satisfying

$$g''-gh'^2=-c_1g'(1+g'^2+(gh')^2),$$
$$gh''+2g'h'=g(1-c_1h')(1+g'^2+(gh')^2),$$
$$f(z)=\frac{1}{\sqrt{g'^2+(gh')^2}},\qquad m(z)=\arccos\frac{gh'}{\sqrt{g'^2+(gh')^2}}.$$

For ω_8, invariants are

$$z=\psi t^{-\frac{1}{2}},\qquad \frac{\sqrt{x^2+\phi^2}}{t},\qquad \arctan\frac{x}{\phi}-\frac{c_2}{2}\ln t,\qquad \frac{v}{\sqrt{1+u^2}},\qquad \arctan\frac{u}{c_1}+\frac{t}{2}.$$

So the group-invariant solutions are given by

$$\phi=t^{\frac{1}{2}}g(z)\cos\lambda,\qquad x=t^{\frac{1}{2}}g(z)\sin\lambda,$$
$$u=-\tan[\lambda+m(z)],\qquad v=f(z)\sec[\lambda+m(z)],$$

where $\lambda=h(z)+\frac{c_1}{2}\ln t$, $f(z)$, $g(z)$, $h(z)$ and $m(z)$ satisfy

$$g''-gh'^2=-(g+zg')(1+g'^2+(gh')^2),$$
$$gh''+2g'h'=-(c_1+zh')(1+g'^2+(gh')^2),$$
$$f=\frac{1}{\sqrt{g'^2+(gh')^2}},\qquad m=-\arcsin\frac{g'}{\sqrt{g'^2+(gh')^2}}.$$

Finally for ω_9, invariants are

$$z = \frac{\psi}{t^{\frac{1}{2}}}, \qquad \sqrt{\frac{(\tilde{x}^2 + \tilde{\phi}^2)}{t}}, \qquad \arctan u + \frac{1}{2}\ln t, \qquad \frac{v}{\sqrt{1+u^2}}, \qquad \arctan\frac{\tilde{\phi}}{\tilde{x}} + \frac{c_1}{2}\ln[\tilde{x}^2 + \tilde{\phi}^2],$$

where $\tilde{x} = x - c_1/(1+c_1{}^2)$, $\tilde{\phi} = \phi + 1/(1+c_1{}^2)$. So the group-invariant solutions are given by

$$\phi = -\frac{1}{1+c_1{}^2} + \sqrt{t}g(z)\sin\lambda, \qquad x = \frac{c_1}{1+c_1{}^2} + \sqrt{t}g(z)\cos\lambda,$$
$$u = \tan[\lambda + m(z)], \qquad v = f(z)\sec[\lambda + m(z)],$$

where $\lambda = h(z) - (c_1/2)\ln t - c_1 \ln g(z)$, and $f(z)$, $g(z)$, $h(z)$ and $m(z)$ satisfy

$$g'' = \frac{1}{2}(g - zg')[1 + g'^2 + (gh' - c_1 g')^2],$$
$$gh'' + g'h' + g' = -\frac{1}{2}zgh'[1 + g'^2 + (gh' - c_1 g')^2],$$
$$f = \frac{1}{\sqrt{g'^2 + (gh' - c_1 g')^2}}, \qquad m = \arcsin\frac{gh' - c_1 g'}{\sqrt{g'^2 - c_1 g'^2}}.$$

4. Potential symmetries of (1.1) with three-component equations

In this section, we consider the system

$$\begin{aligned} u_t &= (f(u,v,w)u_x)_x, \\ v_t &= (g(u,v,w)v_x)_x, \\ w_t &= (h(u,v,w)w_x)_x. \end{aligned} \tag{4.1}$$

Analogous to one- and two-component case, introducing the potential variables ϕ, ψ and ρ, we arrive at the auxiliary system

$$\begin{aligned} u &= \phi_x, & \phi_t &= f(u,v,w)u_x, \\ v &= \psi_x, & \psi_t &= g(u,v,w)v_x, \\ w &= \rho_x, & \rho_t &= h(u,v,w)w_x. \end{aligned} \tag{4.2}$$

Similar to the analysis for the two-component case, we arrive at the following result.

Theorem 4.1. *System (4.1) admits potential symmetries if and only if $f = g = h$ satisfies*

$$f = g = h = \frac{1}{u^2 + v^2 \pm w^2 + a}, \qquad \mathrm{a} = \pm 1, 0, \tag{4.3}$$

or

$$f = g = h = \frac{1}{(u+v+w)^2 + a}, \qquad \mathrm{a} = \pm 1, 0, \tag{4.4}$$

or

$$f = g = h = \frac{1}{u^2 + bv^2 + w}, \qquad \mathrm{b} = \pm 1, \tag{4.5}$$

or

$$f = g = h = \frac{1}{u^2 + v + w}, \tag{4.6}$$

or

$$f = g = h = \frac{1}{u + v + w}, \tag{4.7}$$

or $f = g = h$ satisfies the following equation:

$$
\begin{aligned}
[A + Bu + Cv + Dw - u(au + bv + cw)]f_u &+ [E + Fu + Gv + Hw - v(au + bv + cw)]f_v \\
&+ [J + Ku + Lv + Mw - w(au + bv + cw)]f_w \\
&= (2au + 2bv + 2cw - d)f, \qquad (4.8)
\end{aligned}
$$

up to translations and dilatation for u, v and w. The corresponding vector fields of Lie point symmetries to (4.1) with (4.3)–(4.8) are given as follows. For $f = g = h$ given by (4.3) with $a \neq 0$,

$$
\begin{aligned}
&V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_3 = \partial_\phi, \quad V_4 = \partial_\psi, \quad V_5 = \partial_\rho \\
&V_6 = \phi\partial_x - ax\partial_\phi - uv\partial_v - uw\partial_w - (a + u^2)\partial_u, \\
&V_7 = \psi\partial_x - ax\partial_\psi - uv\partial_u - vw\partial_w - (a + v^2)\partial_v, \\
&V_8 = \rho\partial_x \mp ax\partial_\rho - uw\partial_u - vw\partial_v \mp (a \pm w^2)\partial_w, \\
&V_9 = \psi\partial_\phi - \phi\partial_\psi + v\partial_u - u\partial_v, \\
&V_{10} = \phi\partial_\rho \mp \rho\partial_\phi + u\partial_w \mp w\partial_u, \\
&V_{11} = \psi\partial_\rho \mp \rho\partial_\psi + v\partial_w \mp w\partial_v, \\
&V_{12} = \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho + x\partial_x + 2t\partial_t.
\end{aligned}
$$

For $f = g = h$ given by (4.3) with $a = 0$,

$$
\begin{aligned}
&V_1 = \partial_t, V_2 = \partial_x, \quad V_3 = \partial_\phi, \quad V_4 = \partial_\psi, \quad V_5 = \partial_\rho \\
&X_6 = \phi\partial_x - u(u\partial_u + v\partial_v + w\partial_w), \\
&X_7 = \psi\partial_x - v(u\partial_u + v\partial_v + w\partial_w), \\
&V_8 = \rho\partial_x - w(u\partial_u + v\partial_v + w\partial_w), \\
&V_9 = \psi\partial_\phi - \phi\partial_\psi + v\partial_u - u\partial_v, \\
&V_{10} = \phi\partial_\rho \mp \rho\partial_\phi + u\partial_w \mp w\partial_u, \\
&V_{11} = \psi\partial_\rho \mp \rho\partial_\psi + v\partial_w \mp w\partial_v, \\
&V_{12} = \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho + x\partial_x + 2t\partial_t, \\
&V_{13} = u\partial_u + v\partial_v + w\partial_w - x\partial_x, \\
&V_{14} = (\phi^2 + \psi^2 \pm \rho^2 + 2t)\partial_x - 2(u\phi + v\psi \pm w\rho)(u\partial_u + v\partial_v + w\partial_w).
\end{aligned}
$$

For $f = g = h$ given by (4.4) with $a = 0$,

$$
\begin{aligned}
&V_1 = \partial_t, \quad V_2 = \partial_x, \quad V_3 = \partial_\phi, \quad V_4 = \partial_\psi, \quad V_5 = \partial_\rho \\
&V_6 = x(\partial_\phi - \partial_\rho) + \partial_u - \partial_v, \\
&V_7 = x(\partial_\psi - \partial_\rho) + \partial_v - \partial_w, \\
&V_8 = x\partial_x - u\partial_u - v\partial_v - w\partial_w, \\
&V_9 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho, \\
&V_{10} = h_1(\partial_\phi - \partial_\rho) + \mu h_{1,\lambda}(\partial_u - \partial_w), \\
&V_{11} = h_2(\partial_\psi - \partial_\rho) + \mu h_{2,\lambda}(\partial_v - \partial_w), \\
&V_{12} = h_3\partial_x - \mu h_{3,\lambda}(u\partial_u + v\partial_v + w\partial_w), \\
&V_{13} = t\partial_t - \tfrac{1}{4}\lambda(x\partial_x + \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho) + \left(\tfrac{1}{4}\lambda^2 + \tfrac{1}{2}t\right)\partial_\rho \\
&\qquad + \tfrac{1}{4}\mu[(xu - \phi)\partial_u + (xv - \psi)\partial_v + (xw - \rho)\partial_w],
\end{aligned}
$$

where and hereafter $\lambda = \phi + \psi + \rho$, $\mu = u + v + w$. h_i, $i = 1, 2, 3$ are functions of t and λ, and they satisfy the heat equation $h_{i,t} = h_{i,\lambda\lambda}$, where $h_{i,\lambda}$ denotes the derivative of h_i with respect to λ.

For $f = g = h$ given by (4.4) with $a \neq 0$,

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi, \qquad V_5 = \partial_\rho$$
$$V_6 = x(\partial_\phi - \partial_\rho) + \partial_u - \partial_v,$$
$$V_7 = x(\partial_\psi - \partial_\rho) + \partial_v - \partial_w,$$
$$V_8 = 2t\partial_t + \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho,$$
$$V_9 = \lambda(\partial_\phi - \partial_\rho) + \mu(\partial_u - \partial_w),$$
$$V_{10} = \lambda(\partial_\psi - \partial_\rho) + \mu(\partial_v - \partial_w),$$
$$V_{11} = \lambda\partial_x - \mu(u\partial_u + v\partial_v + w\partial_w) + a\mu\partial_w.$$

For $f = g = h$ given by (4.5),

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi, \qquad V_5 = \partial_\rho$$
$$V_6 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho,$$
$$V_7 = x\partial_\psi - 2b\psi\partial_\rho + \partial_v - 2bv\partial_w,$$
$$V_8 = \psi\partial_\phi - \frac{1}{b}\phi\partial_\psi + v\partial_u - \frac{1}{b}u\partial_v,$$
$$V_9 = x\partial_\phi - 2\phi\partial_\rho - 2u\partial_w + \partial_u,$$
$$V_{10} = -2\phi\partial_x + \rho\partial_\phi + (w + 2u^2)\partial_u + 2uv\partial_v + 2uw\partial_w,$$
$$V_{11} = \psi\partial_x - \frac{1}{2b}\rho\partial_\psi - uv\partial_u - \left(\frac{1}{2b}w + v^2\right)\partial_v - vw\partial_w.$$

For $f = g = h$ given by (4.6),

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi, \qquad V_5 = \partial_\rho$$
$$V_6 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho,$$
$$V_7 = \partial_v - \partial_w + x(\partial_\psi - \partial_\rho),$$
$$V_8 = \partial_u - 2u\partial_v + x\partial_\phi - 2\phi\partial_\psi,$$
$$V_9 = \partial_u - 2u\partial_w + x\partial_\phi - 2\phi\partial_\rho,$$
$$V_{10} = (2u^2 + v + w)\partial_u + 2u(v\partial_v + w\partial_w) + (\psi + \rho)(\partial_\phi + \partial_\psi) + \psi\partial_\rho - 2\phi\partial_x,$$
$$V_{11} = (\rho + \psi + 2\phi u + xv + xw)(\partial_v - \partial_w) + [x(\rho + \psi) + \phi^2 + 2t](\partial_\psi - \partial_\rho).$$
$$V_{12} = v\partial_v - v\partial_w - \psi\partial_\rho,$$
$$V_{13} = u\partial_u - x\partial_x + 2v\partial_v + 2w\partial_w + \psi\partial_\psi + \rho\partial_\rho.$$

For $f = g = h$ given by (4.7),

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi, \qquad V_5 = \partial_\rho$$
$$V_6 = 2t\partial_t + x\partial_x + \phi\partial_\phi + \psi\partial_\psi + \rho\partial_\rho,$$
$$V_7 = t\partial_t + x\partial_x - u\partial_u - v\partial_v - w\partial_w,$$
$$V_8 = x(\partial_\psi - \partial_\rho) + \partial_v - \partial_w,$$
$$V_9 = x(\partial_\phi - \partial_\rho) + \partial_u - \partial_w,$$
$$V_{10} = (x\lambda + 2t)(\partial_\phi - \partial_\rho) + (\lambda + x\mu)(\partial_u - \partial_w),$$
$$V_{11} = (x\lambda + 2t)(\partial_\psi - \partial_\rho) + (\lambda + x\mu)(\partial_v - \partial_w).$$

For $f = g = h$ satisfying (4.8),

$$V_1 = \partial_t, \qquad V_2 = \partial_x, \qquad V_3 = \partial_\phi, \qquad V_4 = \partial_\psi, \qquad V_5 = \partial_\rho,$$
$$V_{12} = \phi\partial_\phi + \psi\partial_\psi + \rho\partial_p + x\partial_x + 2t\partial_t,$$

$$
\begin{aligned}
V_{15} = {} & (a\phi + b\psi + c\rho)\partial_x + dt\partial_t + (Ax + B\phi + C\psi + D\rho)\partial_\phi \\
& + (Ex + F\phi + G\psi + H\rho)\partial_\psi + (Jx + K\phi + L\psi + Mp)\partial_\rho \\
& + [A + Bu + Cv + Dw - u(au + bv + cw)]\partial_u \\
& + [E + Fu + Gv + Hw - v(au + bv + cw)]\partial_v \\
& + [J + Ku + Lv + Mw - w(au + bv + cw)]\partial_w .
\end{aligned}
$$

5. Concluding remarks

We have shown that systems (2.1) and (4.1) admit the potential symmetries for certain coefficient functions. A natural question is that whether they admit other types of potential symmetries. Indeed, there are other different ways to write system (2.1) by introducing different auxiliary variables. For instance, by introducing one potential variable w, system (2.1) can be written as

$$u = w_x, \qquad w_t = f(u, v)u_x, \qquad v_t = (g(u, v)v_x)_x. \tag{5.1}$$

Assume that system (5.1) admits the Lie point symmetries

$$V = \xi\partial_x + \tau\partial_t + \eta_1\partial_u + \eta_2\partial_w + \eta_3\partial_v, \tag{5.2}$$

thus the infinitesimals satisfy the following equations:

$$
\begin{aligned}
& \eta_1\eta_{1,x} - \xi_{xx}g + \xi_t + ug_u\eta_{1,w} = 0, \\
& \eta_3 g_v^2 - g^2\eta_{3,vv} - \eta_1 g g_{uv} - \eta_{3,v} g g_v + \eta_1 g_u g_v - \eta_3 g g_{vv} = 0, \\
& \eta_{1,v} = \eta_{2,v} = \tau_w = \eta_{2,u} = \xi_w = \tau_x = 0, \\
& \xi_u = \eta_{3,u} = \eta_{3,w} = \tau_v = 0, \\
& f(\eta_{2,w} - \tau_t + \xi_x - \eta_{1,u}) - \eta_1 f_u - \eta_3 f_v = 0, \\
& \eta_{2,t} - f(u\eta_{1,w} + \eta_{1,x}) - u\xi_t = 0, \\
& \eta_1 - \eta_{2,x} - u(\eta_{2,w} - \xi_x) = 0, \\
& \eta_1 g_u + \eta_3 g_v + g(\tau_t - 2\xi_x) = 0, \\
& \eta_3(g_u g_v - g g_{uv}) - \eta_1 g g_{uu} - \eta_{1,u} g g_u + \eta_1 g_u^2 = 0, \\
& \eta_{3,t} = \xi_{1,v} = \eta_{3,x} = \xi_{2,u} = 0.
\end{aligned}
$$

It is readily to show that system (2.1) does not admits this kind of potential symmetries.

System (2.1) can also be written as

$$
\begin{aligned}
& u = w_x, \qquad v = h_x, \qquad w_t = f(u, v)u_x, \\
& p_x = w + h, \qquad p_t = F(u, v),
\end{aligned} \tag{5.3}
$$

where f is related to F by $f = F_u$. So system (5.3) is equivalent to a special case of (2.1). Let us assume that system (5.3) admits the Lie point symmetries

$$X = \xi\partial_x + \tau\partial_t + \eta_1\partial_u + \eta_2\partial_w + \eta_3\partial_v + \eta_4\partial_h + \eta_5\partial_p .$$

We obtain the over-determined system for the infinitesimals by using the infinitesimal criterion for invariance of PDEs. A detail analysis shows that system (2.1) does not admit the potential symmetries associated with system (5.3).

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